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Root Multiplicities of the Hyperbolic Kac–Moody Lie Algebra $HA_1^{(1)}$

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INTRODUCTION

Since the notion of generalized Cartan matrix Lie algebras was introduced in [Kac1] and [M], the Kac–Moody theory has developed into one of the most active field of mathematics in the past two decades with a lot of applications to other branches of mathematics such as number theory, modular forms, combinatorics, and mathematical physics. However, most of the research activities have been concentrated on the class of affine Kac–Moody Lie algebras. For the structure of Kac–Moody Lie algebras of indefinite type, very limited information is available.

In [FF] and [KMW], the level 2 root multiplicities were computed for the hyperbolic Kac–Moody Lie algebras $HA_1^{(1)}$ and E_{10}, respectively. In [Kang1] and [Kang2], we developed a homological theory for the graded Lie algebras, and proposed an inductive program to study the structure of Lorentzian Kac–Moody Lie algebras; i.e., Kac–Moody Lie algebras whose Cartan matrix has a Lorentzian signature. As an application, we computed the root multiplicities of the hyperbolic Kac–Moody Lie algebras $HA_1^{(1)}$ and $HA_2^{(2)}$ up to level 3. In [Kang3], we computed the root multiplicities of the hyperbolic Kac–Moody Lie algebra $HA_1^{(1)}$ up to level 4 using the paths as the labelling scheme. In this work, we concentrate on the study of the hyperbolic Kac–Moody Lie algebra $HA_1^{(1)}$ and give a root multiplicity formula up to level 5.

In the first section, we present some of the basic definitions in the Kac–Moody theory, and realized the hyperbolic Kac–Moody Lie algebra $HA_1^{(1)}$ as the minimal graded Lie algebra L with the local part $V \oplus \mathfrak{g} \oplus V'$, where $\mathfrak{g}$ is the affine Kac–Moody Lie algebra $A_1^{(1)}$, V is the basic representation of $\mathfrak{g}$, and V' is the contragredient of V. Thus $L = G/I$, where G is the maximal graded Lie algebra with the local part $V \oplus \mathfrak{g} \oplus V'$, and I is the

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maximal graded ideal of G intersecting the local part trivially. In the second section, we give a root multiplicity formula for the maximal graded Lie algebra G using the Witt formula, and in Section 3, we recall the homological theory of the graded Lie algebras developed in [Kang1] and [Kang2] to determine the structure of the homogeneous subspaces of I up to level 5. In Sections 4 and 5, we used the technique developed in [Fe] to decompose the homogeneous subspaces of I into the direct sum of integrable irreducible highest weight modules over $A_1^{(1)}$. In the final section, we give a root multiplicity formula for the hyperbolic Kac–Moody Lie algebra $HA_1^{(1)}$ up to level 5 using the crystal bases of integrable irreducible highest weight representations of the quantum affine Lie algebra $U_q(\widehat{sl}(2))$ [DJKMO, JMMO]. The extended Young diagrams are used as the labelling scheme.

1. Construction of $HA_1^{(1)}$

We start with the basic definitions in the Kac–Moody theory. An $n \times n$ matrix $A = (a_{ij})$ is called a *generalized Cartan matrix* if it satisfies the following conditions: (i) $a_{ii} = 2$ for $i = 1, ..., n$, (ii) a_{ij} are non-positive integers for $i \neq j$, (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. A is called *symmetrizable* if DA is symmetric for some diagonal matrix $D = \operatorname{diag}(q_1, ..., q_n)$ with $q_i > 0$, $q_i \in \mathbf{Q}$. A *realization* of an $n \times n$ matrix A of rank l is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a $(2n-l)$-dimensional complex vector space, $\Pi = \{\alpha_1, ..., \alpha_n\}$ and $\Pi^\vee = \{\alpha_1^\vee, ..., \alpha_n^\vee\}$ are linearly independent indexed subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$ for $i, j = 1, ..., n$. The *Kac–Moody Lie algebra* $\mathfrak{g}(A)$ associated with a generalized Cartan matrix A is the Lie algebra generated by the elements e_i, f_i $(i = 1, ..., n)$ and $\mathfrak{h}$ with the following defining relations:

$$
\begin{aligned}
&[h, h'] = 0 \quad \text{for} \quad h, h' \in \mathfrak{h},\\
&[e_i, f_j] = \delta_{ij}\alpha_i^\vee \quad \text{for} \quad i, j = 1, ..., n,\\
&[h, e_j] = \alpha_j(h)\, e_j, \quad [h, f_j] = -\alpha_j(h)\, f_j \quad \text{for} \quad j = 1, ..., n, \qquad (1.1)\\
&(\operatorname{ad} e_i)^{1-a_{ij}}(e_j) = 0 \quad \text{for} \quad i, j = 1, ..., n \quad \text{with} \quad i \neq j,\\
&(\operatorname{ad} f_i)^{1-a_{ij}}(f_j) = 0 \quad \text{for} \quad i, j = 1, ..., n \quad \text{with} \quad i \neq j.
\end{aligned}
$$

An indecomposable generalized Cartan matrix A is said to be of *finite* type if all its principal minor are positive, of *affine* type if all its proper principal minors are positive and $\det A = 0$, and of *indefinite* type if A is of neither finite nor affine type. A is of *hyperbolic* type it it is of indefinite type and all its proper principal submatrices are of finite or affine type.

Let $\mathfrak{g}$ be a Lie algebra, and V, V' be modules over $\mathfrak{g}$ with a $\mathfrak{g}$-module homomorphism $\psi: V' \otimes V \to \mathfrak{g}$. Then the space $V \oplus \mathfrak{g} \oplus V'$ has a local Lie algebra structure (see, e.g., [BKM]). Thus there exist maximal and minimal graded Lie algebras $G = G(\mathfrak{g}, V, V', \psi) = \bigoplus_{n \in \mathbf{Z}} G_n$ and $L = L(\mathfrak{g}, V, V', \psi) = \bigoplus_{n \in \mathbf{Z}} L_n$ with the local part $V \oplus \mathfrak{g} \oplus V'$. We define a graded ideal $I = I(\mathfrak{g}, V, V', \psi) = \bigoplus_{n \in \mathbf{Z}} I_n$ as follows. For $n \geqslant 2$, let

$$I_{\pm n} = \{x \in G_{\pm n} \mid (\operatorname{ad} G_{\mp 1})^{n-1} x = 0\},$$

and define $I = \bigoplus_{n \in \mathbf{Z}} I_n$. Set $I_\pm = \bigoplus_{n \geqslant 2} I_{\pm n}$. Then the subspaces I and $I_\pm$ are all graded ideals of G, and I is the maximal graded ideal intersecting the local part trivially [BKM, FF, Kang1]. Hence $L = G/I$. Write $G_\pm = \bigoplus_{n \geqslant 1} G_{\pm n}$ and $L_\pm = \bigoplus_{n \geqslant 1} L_{\pm n}$. Thus $L_\pm = G_\pm / I_\pm$. Note that $G_\pm$ is the free Lie algebra generated by $G_{\pm 1}$. By restricting the choices of $\mathfrak{g}$, V, V', and ψ, we can realize a large family of interesting Lie algebras as the minimal graded Lie algebras L with the local part $V \oplus \mathfrak{g} \oplus V'$. For instance, in [BKM], we construct most of finite dimensional classical simple Lie algebras, Lie algebras of Cartan type, Kostrikin's one parameter family of simple Lie algebras of dimension 10 of characteristic 3, and symmetrizable Kac–Moody Lie algebras.

Now we will realize the hyperbolic Kac–Moody Lie algebra $HA_1^{(1)}$ with Cartan matrix

$$A = (a_{ij})_{i, j = -1, 0, 1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

using the above construction. Let $\mathfrak{g}$ be the affine Kac–Moody Lie algebra $A_1^{(1)}$ with Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, V be the basic representation of $\mathfrak{g}$ with highest weight Λ_0, and V' be the contragredient of V. We define a $\mathfrak{g}$-module homomorphism $\psi: V' \otimes V \to \mathfrak{g}$ by

$$\psi(v' \otimes v) = -\sum_{i \in \Lambda} \langle v', x_i \cdot v \rangle \, y_i,$$

where $\langle \ , \ \rangle$ denotes the bilinear pairing between V' and V, and $\{x_i\}_{i \in \Lambda}$ and $\{y_i\}_{i \in \Lambda}$ are dual bases of $\mathfrak{g}$ with respect to the standard invariant symmetric bilinear form on $\mathfrak{g}$ [Kac 2, Chap. 2]. Let $L = L(\mathfrak{g}, V, V', \psi)$ be the minimal graded Lie algebra with the local part $V \oplus \mathfrak{g} \oplus V'$ as defined above. Then one can prove that L is isomorphic to the hyperbolic Kac–Moody Lie algebra $HA_1^{(1)}$ [FF, Kang1]. We denote by α_{-1}, α_0, α_1 the simple roots of $HA_1^{(1)}$. Thus $V = V(-\alpha_{-1})$ is the irreducible highest weight module over $\mathfrak{g}$ with highest weight $-\alpha_{-1}$ and V' is the irreducible lowest weight module over $\mathfrak{g}$ with lowest weight α_{-1}. Let v_0 and v_0' be the highest and lowest weight vectors of V and V', respectively, with $\langle v_0', v_0 \rangle = 1$. By the Gabber–Kac Theorem [GK], the ideal I is generated by the elements $(\operatorname{ad} v_0)^2 (f_0)$ and $(\operatorname{ad} v_0')^2 (e_0)$. In particular, $I_\pm$ is generated by the subspaces $I_{\pm 2}$. Note that I_2 is isomorphic to the irreducible lowest weight module over $\mathfrak{g}$ with lowest weight $2\alpha_{-1} + \alpha_0$ and that I_{-2} is isomorphic to the irreducible highest weight module over $\mathfrak{g}$ with highest weight $-2\alpha_{-1} - \alpha_0$.

2. The Witt Formula

In this section, we give a root multiplicity formula for the maximal graded Lie algebra G using the following generalization of the Witt formula [Kang1, Kang2].

Theorem 2.1. *Let $V = V(\Lambda)$ be an integrable module over a symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$ and let $F = \bigoplus_{n \geq 1} F_n$ be the free Lie algebra generated by V. Then F is an integrable graded module over $\mathfrak{g}$. Let $S = \{\tau_i \mid i = 1, 2, 3, \ldots\}$ be an enumeration of all the weights of V. For a weight τ of F, set*

$$T(\tau, V) = \left\{ (n) = (n_1, n_2, n_3, \ldots) \mid n_i \in \mathbf{Z}_{\geq 0}, \sum n_i \tau_i = \tau \right\}, \tag{2.1}$$

and define

$$B(\tau, V) = \sum_{(n) \in T(\tau)} \frac{((\sum n_i) - 1)!}{\prod (n_i!)} \prod (\dim V_{\tau_i})^{n_i}. \tag{2.2}$$

Then we have

$$\dim F_\alpha = \sum_{\tau \mid \alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B(\tau, V), \tag{2.3}$$

where $\tau \mid \alpha$ if $\alpha = k\tau$ for some positive integer k, in which case $\alpha/\tau = k$ and $\tau/\alpha = 1/k$.

Remark 2.2. We call the function $B(\tau, V)$ the *Witt partition function* on V. If $\tau = k\Lambda + \sum_i n_i \alpha_i$, then the set $T(\tau, V)$ corresponds to the partition of τ into k parts. The formula (2.3) is called the *Witt formula.*

We consider the maximal graded Lie algebra G with the local part $V \oplus \mathfrak{g} \oplus V'$. We know that $\dim (G_0)_\alpha = \dim \mathfrak{g}_\alpha = 1$ for all roots α and of G_0 (e.g., [Kac2]). Since $\dim G_\alpha = \dim G_{-\alpha}$ for all α, we will consider the negative part only. In Section 1, we have seen that G_- is the free Lie algebra generated by the space $G_{-1} = V$. The roots of G_{-1} are of the form $-\alpha_{-1} - k\alpha_0 - l\alpha_1$, where k and l are nonnegative integers satisfying the inequality $k-(k-l)^2 \geqslant 0$ (e.g., [FK, Kac2]. We enumerate the roots of G_{-1} lexicographically:

$$\begin{aligned}
\tau_1 &= -\alpha_{-1}, \\
\tau_2 &= -\alpha_{-1} - \alpha_0, \\
\tau_3 &= -\alpha_{-1} - \alpha_0 - \alpha_1, \\
\tau_4 &= -\alpha_{-1} - \alpha_0 - 2\alpha_1, \\
\tau_5 &= -\alpha_{-1} - 2\alpha_0 - \alpha_1, \\
\tau_6 &= -\alpha_{-1} - 2\alpha_0 - 2\alpha_1, \\
\tau_7 &= -\alpha_{-1} - 2\alpha_0 - 3\alpha_1, \\
\tau_8 &= -\alpha_{-1} - 3\alpha_0 - 2\alpha_1, \\
&\;\vdots
\end{aligned} \tag{2.4}$$

Let $(\ |\ )$ denote the standard invariant symmetric bilinear form defined on $\mathfrak{g}$. For a root $\alpha = -\alpha_{-1} - k\alpha_0 - l\alpha_1$ of G_{-1}, we have by [FL],

$$\dim(G_{-1})_\alpha = p\left(1 - \frac{(\alpha|\alpha)}{2}\right) = p(k-(k-l)^2), \tag{2.5}$$

where the function $p(n)$ is defined by

$$\sum_{n \geqslant 0} p(n)\, q^n = \frac{1}{\phi(q)} = \frac{1}{\prod_{n \geqslant 1} (1-q^n)}. \tag{2.6}$$

Thus the Witt partition function becomes

$$B(\tau, G_{-1}) = \sum_{(n) \in T(\tau)} \frac{((\sum n_i) - 1)!}{\prod (n_i!)} \prod p\left(1 - \frac{(\tau_i|\tau_i)}{2}\right)^{n_i}. \tag{2.7}$$

Therefore the Witt formula gives the root multiplicities of G:

$$\dim G_\alpha = \sum_{\tau | \alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B(\tau, G_{-1}). \tag{2.8}$$

Example 2.3. We illustrate the formula (2.8) for the root $\alpha = -3\alpha_{-1} - 4\alpha_0 - 4\alpha_1$. Since the only root that can divide α is itself, we have

$$\dim G_\alpha = B(\alpha, G_{-1}).$$

Using the lexicographical ordering, the partitions of α into 3 parts are given in the following:

$$
\begin{aligned}
&-3\alpha_{-1}-4\alpha_0-4\alpha_1\\
&\quad=(-\alpha_{-1})+(-\alpha_{-1})+(-\alpha_{-1}-4\alpha_0-4\alpha_1)\\
&\quad=(-\alpha_{-1})+(-\alpha_{-1}-\alpha_0)+(-\alpha_{-1}-3\alpha_0-4\alpha_1)\\
&\quad=(-\alpha_{-1})+(-\alpha_{-1}-\alpha_0-\alpha_1)+(-\alpha_{-1}-3\alpha_0-3\alpha_1)\\
&\quad=(-\alpha_{-1})+(-\alpha_{-1}-\alpha_0-2\alpha_1)+(-\alpha_{-1}-3\alpha_0-2\alpha_1)\\
&\quad=(-\alpha_{-1})+(-\alpha_{-1}-2\alpha_0-\alpha_1)+(-\alpha_{-1}-2\alpha_0-3\alpha_1)\\
&\quad=(-\alpha_{-1})+(-\alpha_{-1}-2\alpha_0-2\alpha_1)+(-\alpha_{-1}-2\alpha_0-2\alpha_1)\\
&\quad=(-\alpha_{-1}-\alpha_0)+(-\alpha_{-1}-\alpha_0-\alpha_1)+(-\alpha_{-1}-2\alpha_0-3\alpha_1)\\
&\quad=(-\alpha_{-1}-\alpha_0)+(-\alpha_{-1}-\alpha_0-2\alpha_1)+(-\alpha_{-1}-2\alpha_0-2\alpha_1)\\
&\quad=(-\alpha_{-1}-\alpha_0-\alpha_1)+(-\alpha_{-1}-\alpha_0-\alpha_1)+(-\alpha_{-1}-2\alpha_0-2\alpha_1)\\
&\quad=(-\alpha_{-1}-\alpha_0-\alpha_1)+(-\alpha_{-1}-\alpha_0-2\alpha_1)+(-\alpha_{-1}-2\alpha_0-\alpha_1).
\end{aligned}
$$

Therefore

$$B(\alpha, G_{-1})=5+4+6+4+2+4+2+4+2+2=35.$$

Hence

$$\dim G_{-3\alpha_{-1}-4\alpha_0-4\alpha_1}=35.$$

Similarly, we have

$$\dim G_{-4\alpha_{-1}-5\alpha_0-4\alpha_1}=150,$$

and

$$\dim G_{-5\alpha_{-1}-6\alpha_0-5\alpha_1}=926.$$

The Witt formula can also be used to give a weight multiplicity formula for the symmetric and antisymmetric squares of integrable modules over a symmetrizable Kac–Moody Lie algebra. The following proposition is due to G. B. Seligman.

PROPOSITION 2.4. *Let V be an integrable module over a symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$. Then we have*

$$\dim S^2(V)_\alpha = B(\alpha, V)+\tfrac{1}{2}\varepsilon(\alpha, 2)\dim V_{\alpha/2}, \tag{2.9}$$

$$\dim \Lambda^2(V)_\alpha = B(\alpha, V)-\tfrac{1}{2}\varepsilon(\alpha, 2)\dim V_{\alpha/2}, \tag{2.10}$$

and hence

$$\dim(V \otimes V)_\alpha = 2B(\alpha, V), \tag{2.11}$$

where the function $\varepsilon(\alpha, t)$ *is defined by*

$$\begin{aligned} \varepsilon(\alpha, t) &= 1 \qquad \text{if } \alpha \text{ is divisible by } t \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{2.12}$$

EXAMPLE 2.5. In this example, we will compute $\dim \Lambda^2(I_{-2})_\alpha$ for $\alpha = -4\alpha_{-1} - 5\alpha_0 - 4\alpha_1$. Since $\varepsilon(\alpha, 2) = 0$ and $I_{-2} \cong V(-2\alpha_{-1} - \alpha_0)$, (2.10) gives

$$\dim \Lambda^2(I_{-2})_\alpha = B(\alpha, V(-2\alpha_{-1} - \alpha_0)).$$

The weights of I_{-2} are of the form $-2\alpha_{-1} - k\alpha_0 - l\alpha_1$, where k and l are nonnegative integers satisfying the inequality $2k - (k-l)^2 \geqslant 1$. By [FL], we have

$$\dim(I_{-2})_\alpha = E\left(3 - \frac{(\alpha \mid \alpha)}{2}\right) = E(2k - (k-l)^2 - 1),$$

where the function $E(n)$ is defined by

$$\sum_{n \geqslant 0} E(n)\, q^n = \frac{\phi(q^2)}{\phi(q)\,\phi(q^4)}. \tag{2.13}$$

We enumerate the weights of I_{-2} lexicographically:

$$\begin{aligned} \tau_1 &= -2\alpha_{-1} - \alpha_0, \\ \tau_2 &= -2\alpha_{-1} - \alpha_0 - \alpha_1, \\ \tau_3 &= -2\alpha_{-1} - \alpha_0 - 2\alpha_1, \\ \tau_4 &= -2\alpha_{-1} - 2\alpha_0 - \alpha_1, \\ \tau_5 &= -2\alpha_{-1} - 2\alpha_0 - 2\alpha_1, \\ \tau_6 &= -2\alpha_{-1} - 2\alpha_0 - 3\alpha_1, \\ \tau_7 &= -2\alpha_{-1} - 3\alpha_0 - \alpha_1, \\ \tau_8 &= -2\alpha_{-1} - 3\alpha_0 - 2\alpha_1, \\ &\ \vdots \end{aligned}$$

Then the partition of α into 2 parts are

$$\begin{aligned}-4\alpha_{-1}-5\alpha_0-4\alpha_1 &= (-2\alpha_{-1}-\alpha_0)+(-2\alpha_{-1}-4\alpha_0-4\alpha_1)\\ &= (-2\alpha_{-1}-\alpha_0-\alpha_1)+(-2\alpha_{-1}-4\alpha_0-3\alpha_1)\\ &= (-2\alpha_{-1}-\alpha_0-2\alpha_1)+(-2\alpha_{-1}-4\alpha_0-2\alpha_1)\\ &= (-2\alpha_{-1}-2\alpha_0-\alpha_1)+(-2\alpha_{-1}-3\alpha_0-3\alpha_1)\\ &= (-2\alpha_{-1}-2\alpha_0-2\alpha_1)+(-2\alpha_{-1}-3\alpha_0-2\alpha_1)\\ &= (-2\alpha_{-1}-2\alpha_0-3\alpha_1)+(-2\alpha_{-1}-3\alpha_0-\alpha_1).\end{aligned}$$

Therefore

$$B(\alpha, I_{-2}) = 7+5+2+4+6+1 = 25.$$

Hence

$$\dim \Lambda^2(I_{-2})_\alpha = 25.$$

3. Homological Approach

We briefly recall the homological theory for the hyperbolic Kac–Moody Lie algebra $HA_1^{(1)}$ developed in [Kang1] and [Kang2]. Let $I_-^{(j)} = \sum_{n\geq j} I_{-n}$ for $j \geq 2$. Since the subspace I_{-2} generates the ideal I_- of G_-, it is easy to see that $I_-^{(j)}$ is a graded ideal of G_- generated by the subspace I_{-j}. Consider the quotient Lie algebra $L_-^{(j)} = G_-/I_-^{(j)}$ and let $J_-^{(j)} = I_-^{(j)}/[I_-^{(j)}, I_-^{(j)}]$. Then $J_-^{(j)}$ is an $L_-^{(j)}$-module generated by the subspace J_{-j} which is isomorphic to I_{-j} as modules over $\mathfrak{g}$. Thus we have the following exact sequence of $L_-^{(j)}$-modules

$$0 \longrightarrow K^{(j)} \longrightarrow U(L_-^{(j)}) \otimes J_{-j} \xrightarrow{\psi} J_-^{(j)} \longrightarrow 0,$$

where ψ is the usual Lie bracket mapping, and $K^{(j)}$ is the kernel of ψ. From the long exact sequence

$$\begin{aligned}\cdots &\to H_1(L_-^{(j)}, K^{(j)}) \to H_1(L_-^{(j)}, U(L_-^{(j)})\otimes J_{-j}) \to H_1(L_-^{(j)}, J_-^{(j)})\\ &\to H_0(L_-^{(j)}, K^{(j)}) \to H_0(L_-^{(j)}, U(L_-^{(j)})\otimes J_{-j}) \to H_0(L_-^{(j)}, J_-^{(j)}) \to 0,\end{aligned}$$

we get a $\mathfrak{g}$-module isomorphism

$$H_1(L_-^{(j)}, J_-^{(j)}) \cong H_0(L_-^{(j)}, K^{(j)}) \cong K^{(j)}/L_-^{(j)} \cdot K^{(j)}. \tag{3.1}$$

On the other hand, by the Poincaré–Birkhoff–Witt theorem, we have the following exact sequence of $L^{(j)}$-modules (e.g. [Exercise 9.13, Kac2], [Lemma 4.5, Kang1]):

$$0 \to J^{(j)} \to U(L^{(j)}) \otimes V \to U(L^{(j)}) \to \mathbf{C} \to 0. \tag{3.2}$$

In [Kang1] and [Kang2], we proved that there is an isomorphism of $\mathfrak{g}$-modules

$$H_1(L^{(j)}, J^{(j)}) \cong H_3(L^{(j)}). \tag{3.3}$$

Therefore combining this with (3.1) yields

$$K^{(j)}/L^{(j)} \cdot K^{(j)} \cong H_3(L^{(j)}). \tag{3.4}$$

In particular,

$$H_3(L^{(j)})_{-(j+1)} \cong K^{(j)}_{-(j+1)}.$$

It follows that

$$I_{-(j+1)} \cong (V \otimes I_{-j})/H_3(L^{(j)})_{-(j+1)}. \tag{3.5}$$

By the Kostant formula and Hochschild–Serre spectral sequences ([Liu], [Kang1], [Kang2]), we have

$$H_3(L^{(2)})_{-3} = 0, \tag{3.6}$$

$$H_3(L^{(3)})_{-4} = V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2}), \tag{3.7}$$

$$H_3(L^{(4)})_{-5} = V \otimes \Lambda^2(I_{-2}). \tag{3.8}$$

Hence we have:

PROPOSITION 3.1.

$$I_{-3} \cong V \otimes I_{-2}, \tag{3.9}$$

$$I_{-4} \cong V \otimes I_{-3}/(V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2})), \tag{3.10}$$

$$I_{-5} \cong V \otimes I_{-4}/V \otimes \Lambda^2(I_{-2}). \tag{3.11}$$

4. TENSOR PRODUCT OF MODULES OVER $A_1^{(1)}$

Let $V(\lambda)$ and $V(\tau)$ be two integrable irreducible highest weight modules over a symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$. Then $V(\lambda) \otimes V(\tau)$ is also integrable and hence completely reducible ([Corollary 10.7, Kac2]). So we have the decomposition

$$V(\lambda)\otimes V(\tau)=\bigoplus_{\nu} c_{\lambda,\tau}^{\nu}\, V(\nu).$$

The number $c_{\lambda,\tau}^{\nu}$ is called the outer multiplicity of $V(\nu)$ in $V(\lambda)\otimes L(\tau)$. We know that each ν has the form $\nu=\mu+\tau$, where μ is a weight of $V(\lambda)$ (e.g. [Fe]). Thus we may rewrite the decomposition

$$V(\lambda)\otimes V(\tau)=\bigoplus_{\mu\in P(\lambda)} c_{\lambda,\tau}^{\mu+\tau} V(\mu+\tau),$$

where $P(\lambda)$ denotes the set of all the weights of $V(\lambda)$.

In [Fe], Feingold generalized the Racah formula to the case of symmetrizable Kac–Moody Lie algebras to obtain

$$c_{\lambda,\tau}^{\mu+\tau}=\sum_{w\in W}(\det w)\,\mathrm{mult}_{V(\lambda)}(\mu+\tau+\rho-w(\tau+\rho)).$$

Thus to compute the outer multiplicities, it is essential to have a formula for $\tau+\rho-w(\tau+\rho)$. For the cases of $A_1^{(1)}$ and $A_2^{(2)}$, one can find explicit formulas for them in [LM]. For our purpose, we need the formula for $A_1^{(1)}$ only.

PROPOSITION 4.1 [LM]. *Let $n_0, n_1\in\mathbf{Z}_{\geqslant 0}$ and $j\in\mathbf{Z}$. Let $\tau=n_0\Lambda_0+n_1\Lambda_1$ be a dominant integral weight of $A_1^{(1)}$. If $w=(r_0r_1)^j$, then*

$$\begin{aligned}\tau+\rho-w(\tau+\rho)=&\,j(j(n_0+n_1+2)+n_1+1)\,\alpha_0\\ &+j(j(n_0+n_1+2)-n_0-1)\,\alpha_1,\end{aligned}$$

and if $w=r_1(r_0r_1)^j$, then

$$\begin{aligned}\tau+\rho-w(\tau+\rho)=&\,j(j(n_0+n_1+2)+n_1+1)\,\alpha_0\\ &+(j+1)(j(n_0+n_1+2)+n_1+1)\,\alpha_1.\end{aligned}$$

In [Fe], Feingold used the Racah formula and Proposition 4.1 to find an explicit decomposition of the tensor product $V(\lambda)\otimes V(\tau)$ of integrable irreducible modules over $A_1^{(1)}$ when $\lambda=\Lambda_0$ or $\lambda=\rho$. We recall his result for $\lambda=\Lambda_0$ in the following proposition.

PROPOSITION 4.2 [Fe]. *Let $V(\Lambda_0)$ be the basic representation of $A_1^{(1)}$. Let $\tau=n_0\Lambda_0+n_1\Lambda_1$ be a dominant integral weight and let $\mu=\Lambda_0-k_0\alpha_0-k_1\alpha_1$ be a weight of $V(\Lambda_0)$ such that $\mu+\tau$ is a dominant integral weight. Then we have*

$$c^{\mu+\tau}_{\Lambda_0,\tau} = \sum_{j\in\mathbf{Z}} p\{k_0-(k_0-k_1)^2-j(n_1+1)$$
$$-j(n_0+n_1+2)(j(n_0+n_1+3)-2(k_0-k_1))\}$$
$$-\sum_{j\in\mathbf{Z}} p\{k_0-(k_0-k_1)^2-(j(n_0+n_1+2)+n_1+1)$$
$$\times(j(n_0+n_1+3)+2(k_0-k_1)+n_1+1)\},$$

where the function $p(n)$ is defined by (2.5).

By a similar argument, we can prove the following proposition.

PROPOSITION 4.3. *Let $V(2\Lambda_1)$ be the level 2 integrable irreducible highest weight module over $A_1^{(1)}$ with highest weight $2\Lambda_1$. Let $\tau = n_0\Lambda_0 + n_1\Lambda_1$ be a dominant integral weight and let $\mu = 2\Lambda_1 - k_0\alpha_0 - k_1\alpha_1$ be a weight of $V(2\Lambda_1)$ such that $\mu+\tau$ is a dominant integral weight. Then we have*

$$c^{\mu+\tau}_{2\Lambda_1,\tau} = \sum_{j\in\mathbf{Z}} E\{2k_1-(k_0-k_1)^2+2j(n_0+1)$$
$$-j(n_0+n_1+2)(j(n_0+n_1+4)-2(k_0-k_1))\}$$
$$-\sum_{j\in\mathbf{Z}} E\{2k_1-(k_0-k_1)^2-(j(n_0+n_1+2)+n_1+1)$$
$$\times j(n_0+n_1+4)+2(k_0-k_1)+n_1+3)\},$$

where the function $E(n)$ is defined by (2.13).

5. THE STRUCTURE OF THE MAXIMAL GRADED IDEAL

In this section, we will decompose the homogeneous subspaces of the maximal graded ideal I into the direct sum of integrable irreducible highest weight modules over $A_1^{(1)}$ up to level 5, and compute explicitly the outer multiplicity of each irreducible component. Recall that

$$I_{-2} \cong V(-2\alpha_{-1}-\alpha_0).$$

By (3.9),

$$I_{-3} \cong V \otimes I_{-2} \cong V(-\alpha_{-1}) \otimes V(-2\alpha_{-1}-\alpha_0).$$

Since $V(-\alpha_{-1}) = V(\Lambda_0)$ and $V(-2\alpha_{-1}-\alpha_0) = V(2\Lambda_1)$, we have by Proposition 4.2 that

$$I_{-3} \cong \sum_{m\geqslant 0} \{A_m V(-3\alpha_{-1}-\alpha_0-m\delta) \oplus B_m V(-3\alpha_{-1}-(m+1)\delta)\}, \tag{5.1}$$

where

$$A_m = \sum_{j \in \mathbf{Z}} \{p(m - j(20j+3)) - p(m - (4j+3)(5j+3))\},$$

and

$$B_m = \sum_{j \in \mathbf{Z}} \{p(m - (20j^2 + 11j + 1)) - p(m - (20j^2 + 19j + 4))\}.$$

Note that

$$\sum_{m \geq 0} A_m q^m = \sum_{j \in \mathbf{Z}} (q^{j(20j+3)} - q^{(4j+3)(5j+3)}) \left(\sum_{n \geq 0} p(n)\, q^n \right).$$

Replacing q by q^2 yields

$$\sum_{m \geq 0} A_m q^{2m} = \sum_{j \in \mathbf{Z}} (q^{2j(20j+3)} - q^{2(4j+3)(5j+3)}) \left(\sum_{n \geq 0} p(n)\, q^{2n} \right)$$
$$= \left(\sum_{k \equiv 0,\, 3 \pmod 4} (-1)^k\, q^{k(5k+3)/2} \right) \frac{1}{\phi(q^2)},$$

where

$$\phi(q) = \prod_{n \geq 1} (1 - q^n).$$

Similarly,

$$\sum_{m \geq 0} B_m q^{2m+1} = \left(\sum_{k \equiv 0,\, 1 \pmod 4} (-1)^{k+1}\, q^{k(5k+1)/2} \right) \frac{1}{\phi(q^2)}.$$

Let

$$\sum_{m \geq 0} S_m q^n = \left(\sum_{k \in \mathbf{Z}} (-1)^k\, q^{k(5k+3)/2} \right) \frac{1}{\phi(q^2)},$$

and

$$\sum_{m \geq 0} T_m q^m = \left(\sum_{k \in \mathbf{Z}} (-1)^k\, q^{k(5k+1)/2} \right) \frac{1}{\phi(q^2)}.$$

Then $A_m = S_{2m}$ and $B_m = -T_{2m+1}$. It is well known that

$$\sum_{k \in \mathbf{Z}} (-1)^k\, q^{k(5k+3)/2} = \phi(q^5) \prod_{n \equiv \pm 1 \pmod 5} (1 - q^n)$$

and that

$$\sum_{k \in \mathbf{Z}} (-1)^k q^{k(5k+1)/2} = \phi(q^5) \prod_{n \equiv \pm 2 \pmod 5} (1-q^n).$$

Thus we have

$$\sum_{m \geq 0} S_m q^m = \frac{\phi(q^5)}{\phi(q^2)} \prod_{n \equiv \pm 1 \pmod 5} (1-q^n), \tag{5.2}$$

and

$$\sum_{m \geq 0} T_m q^m = \frac{\phi(q^5)}{\phi(q^2)} \prod_{n \equiv \pm 2 \pmod 5} (1-q^n). \tag{5.3}$$

The structure of I_{-3} is given in the following Proposition.

PROPOSITION 5.1. *Let*

$$\sum_{m \geq 0} A_m q^m = \sum_{m \geq 0} S_{2m} q^m$$

and

$$\sum_{m \geq 0} B_m q^m = -\sum_{m \geq 0} T_{2m+1} q^m,$$

where S_m and T_m are defined by (5.2) *and* (5.3). *Then we have*

$$I_{-3} \cong \sum_{m \geq 0} \{A_m V(-3\alpha_{-1} - \alpha_0 - m\delta) \oplus B_m V(-3\alpha_{-1} - (m+1)\delta)\}.$$

Next, we consider the subspace I_{-4}. By (3.10), we have

$$I_{-4} \cong V \otimes I_{-3} / (V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2})).$$

In [Fe], Feingold showed that

$$V \otimes V \cong \sum_{m \geq 0} (K_{2m} V(-2\alpha_{-1} - m\delta) \oplus K_{2m+1} V(-2\alpha_{-1} - \alpha_0 - m\delta)), \tag{5.4}$$

where the coefficients K_m are defined by

$$\sum_{m \geq 0} K_m q^m = \frac{\phi(q^2)^2}{\phi(q)\,\phi(q^4)}. \tag{5.5}$$

Let

$$M = \sum_{m \geq 0} (K_{2m} V(-2\alpha_{-1} - m\delta) \oplus K_{2m+3} V(-2\alpha_{-1} - \alpha_0 - (m+1)\delta)).$$

Then $V \otimes V \cong M \oplus I_{-2}$, and hence

$$\begin{aligned} I_{-4} &\cong V \otimes V \otimes I_{-2}/(V(-4\alpha_{-1}-3\alpha_0-\alpha_1) \oplus S^2(I_{-2})) \\ &\cong (M \oplus I_{-2}) \otimes I_{-2}/(V(-4\alpha_{-1}-3\alpha_0-\alpha_1) \oplus S^2(I_{-2})) \\ &\cong (M \otimes I_{-2}) \oplus (I_{-2} \otimes I_{-2})/(V(-4\alpha_{-1}-3\alpha_0-\alpha_1) \oplus S^2(I_{-2})) \\ &\cong (M \otimes I_{-2}/V(-4\alpha_{-1}-3\alpha_0-\alpha_1)) \oplus \Lambda^2(I_{-2}). \end{aligned}$$

By Proposition 4.3, we have

$$\begin{aligned} M \otimes I_{-2} \cong \sum_{m \geq 0} & \{K_{2m} V(-2\alpha_{-1}-m\delta) \otimes V(-2\alpha_{-1}-\alpha_0) \\ & \oplus K_{2m+3} V(-2\alpha_{-1}-\alpha_0-(m+1)\delta) \otimes V(-2\alpha_{-1}-\alpha_0)\} \\ \cong \sum_{m \geq 0} & \Big\{ K_{2m} \sum_{n \geq 0} \{A'_n V(-4\alpha_{-1}-\alpha_0-(m+n)\delta) \\ & \oplus B'_n V(-4\alpha_{-1}-2\alpha_0-(m+n)\delta) \\ & \oplus C'_n V(-4\alpha_{-1}-(m+n+1)\delta)\} \\ & \oplus K_{2m+3} \sum_{n \geq 0} \{D'_n V(-4\alpha_{-1}-2\alpha_0-(m+n+1)\delta) \\ & \oplus E'_n V(-4\alpha_{-1}-\alpha_0-(m+n+2)\delta) \\ & \oplus F'_n V(-(m+n+3)\delta)\} \Big\}, \end{aligned}$$

where

$$\begin{aligned} A'_n &= \sum_{j \in \mathbf{Z}} \{E(2n-6j(4j-1)) - E(2n-3(2j+1)(4j+1))\}, \\ B'_n &= \sum_{j \in \mathbf{Z}} \{E(2n+1-2(3j+1)(4j+1)) - E(2n+1-(2j+1)(12j+7))\}, \\ C'_n &= \sum_{j \in \mathbf{Z}} \{E(2n+1-2j(12j+1)) - E(2n+1-(4j+1)(6j+1))\}, \\ D'_n &= \sum_{j \in \mathbf{Z}} \{E(2n-2j(12j-1)) - E(2n-(4j+3)(6j+5))\}, \\ E'_n &= \sum_{j \in \mathbf{Z}} \{E(2n+1-6j(4j+1)) - E(2n+1-3(2j+1)(4j+3))\}, \\ F'_n &= \sum_{j \in \mathbf{Z}} \{E(2n+2-2(3j+1)(4j+1)) - E(2n+2-(2j+1)(12j+5))\}. \end{aligned}$$

Thus

$$\sum_{n \geq 0} (A'_n q^{2n} + E'_n q^{2n+1})$$

$$= \sum_{j \in \mathbf{Z}} (q^{6j(4j-1)} - q^{3(2j+1)(4j+1)}) \left(\sum_{m \geq 0} E(2m) q^{2m} \right)$$

$$+ \sum_{j \in \mathbf{Z}} (q^{6j(4j+1)} - q^{3(2j+1)(4j+3)}) \left(\sum_{m \geq 0} E(2m+1) q^{2m+1} \right)$$

$$= \left(\sum_{k \in \mathbf{Z}} (-1)^k q^{3k(2k+1)} \right) \left(\sum_{m \geq 0} E(m) q^m \right).$$

Since

$$\sum_{k \in \mathbf{Z}} (-1)^k q^{3k(2k+1)} = \frac{\phi(q^3)\,\phi(q^{12})}{\phi(q^6)},$$

it follows that

$$\sum_{n \geq 0} (A'_n q^{2n} + E'_n q^{2n+1}) = \frac{\phi(q^2)\,\phi(q^3)\,\phi(q^{12})}{\phi(q)\,\phi(q^4)\,\phi(q^6)}.$$

Let

$$\sum_{n \geq 0} \tilde{A}_n q^n = \frac{\phi(q^2)\,\phi(q^3)\,\phi(q^{12})}{\phi(q)\,\phi(q^4)\,\phi(q^6)} \tag{5.6}$$

Then $A'_n = \tilde{A}_{2n}$ and $E'_n = \tilde{A}_{2n+1}$.

Similarly, we have

$$\sum_{n \geq 0} (D'_n q^{2n} + C'_n q^{2n+1}) = \left(\sum_{k \equiv 0,\,3 \pmod 4} (-1)^k q^{k(3k+1)/2} \right) \frac{\phi(q^2)}{\phi(q)\,\phi(q^4)}, \tag{5.7}$$

and

$$\sum_{n \geq 0} (F'_{n-1} q^{2n} + B'_n q^{2n+1}) = \left(\sum_{k \equiv 1,\,2 \pmod 4} (-1)^{k+1} q^{k(3k+1)/2} \right) \frac{\phi(q^2)}{\phi(q)\,\phi(q^4)}. \tag{5.8}$$

Thus

$$\sum_{n \geq 0} (D'_n q^{2n} + C'_n q^{2n+1}) - \sum_{n \geq 0} (F'_{n-1} q^{2n} + B'_n q^{2n+1})$$

$$= \sum_{n \geq 0} ((D'_n - F'_{n-1})\, q^{2n} + (C'_n - B'_n)\, q^{2n+1})$$

$$= \left(\sum_{k \in \mathbf{Z}} (-1)^k q^{k(3k+1)/2} \right) \frac{\phi(q^2)}{\phi(q)\,\phi(q^4)}.$$

Since

$$\sum_{k \in \mathbf{Z}} (-1)^k q^{k(3k+1)/2} = \phi(q),$$

we get

$$\sum_{m \geqslant 0} \{D'_n - F'_{n-1}) q^{2n} + (C'_n - B'_n) q^{2n+1}\} = \frac{\phi(q^2)}{\phi(q^4)}. \tag{5.9}$$

It follows that $B'_n = C'_n$. We rewrite (5.7) and (5.8):

$$\sum_{n \geqslant 0} (D'_n q^{2n} + B'_n q^{2n+1}) = \left(\sum_{k \equiv 0,\, 3 \pmod 4} (-1)^k q^{k(3k+1)/3} \right) \frac{\phi(q^2)}{\phi(q)\, \phi(q^4)}, \tag{5.10}$$

$$\sum_{n \geqslant 0} (F'_{n-1} q^{2n} + C'_n q^{2n+1}) = \left(\sum_{k \equiv 1,\, 2 \pmod 4} (-1)^{k+1} q^{k(3k+1)/2} \right) \frac{\phi(q^2)}{\phi(q)\, \phi(q^4)}. \tag{5.11}$$

Then (5.9) becomes

$$\sum_{n \geqslant 0} (D'_n q^{2n} + B_n q^{2n+1}) - \sum_{n \geqslant 0} (F'_{n-1} q^{2n} + C_n q^{2n+1}) = \frac{\phi(q^2)}{\phi(q^4)}. \tag{5.12}$$

Replacing q by $-q$ in (5.10) and (5.11) yields

$$\sum_{n \geqslant 0} (D'_n q^{2n} - B'_n q^{2n+1}) = \left(\sum_{k \equiv 0,\, 3 \pmod 4} q^{k(3k+1)/2} \right) \frac{\phi(q)}{\phi(q^2)^2}, \tag{5.13}$$

and

$$\sum_{n \geqslant 0} (F'_{n-1} q^{2n} - C'_n q^{2n+1}) = \left(\sum_{k \equiv 1,\, 2 \pmod 4} q^{k(3k+1)/2} \right) \frac{\phi(q)}{\phi(q^2)^2}. \tag{5.14}$$

It follows that

$$\sum_{n \geqslant 0} \{(D'_n + F'_{n-1}) q^{2n} - (B'_n + C'_n) q^{2n+1}\} = \left(\sum_{k \in \mathbf{Z}} q^{k(3k+1)/2} \right) \frac{\phi(q)}{\phi(q^2)^2}.$$

Since

$$\sum_{k \in \mathbf{Z}} q^{k(3k+1)/2} = \frac{\phi(q^3)^2\, \phi(q^2)}{\phi(q)\, \phi(q^6)},$$

we obtain

$$\sum_{n \geqslant 0} \{(D'_n + F'_{n-1}) q^{2n} - (B'_n + C'_n) q^{2n+1}\} = \frac{\phi(q^3)^2\, \phi(q^2)}{\phi(q)\, \phi(q^6)}.$$

Replacing q by $-q$ yields

$$\sum_{n\geq 0}(D'_n q^{2n}+B'_n q^{2n+1})+\sum_{n\geq 0}(F'_{n-1}q^{2n}+C'_n q^{2n+1})$$
$$=\frac{\phi(q^6)^5}{\phi(q^2)\,\phi(q^3)^2\,\phi(q^{12})^2}. \tag{5.15}$$

Combining (5.12) and (5.15) gives

$$\sum_{n\geq 0}(D'_n q^{2n}+B'_n q^{2n+1})=\frac{1}{2}\left(\frac{\phi(q^2)}{\phi(q^4)}+\frac{\phi(q^6)^5}{\phi(q^2)\,\phi(q^3)^2\,\phi(q^{12})^2}\right),$$

and

$$\sum_{n\geq 0}(F'_{n-1}q^{2n}+C'_n q^{2n+1})=\frac{1}{2}\left(-\frac{\phi(q^2)}{\phi(q^4)}+\frac{\phi(q^6)^5}{\phi(q^2)\,\phi(q^3)^2\,\phi(q^{12})^2}\right).$$

Let

$$\sum_{n\geq 0}\tilde{B}_n q^n=\frac{1}{2}\left(\frac{\phi(q^2)}{\phi(q^4)}+\frac{\phi(q^6)^5}{\phi(q^2)\,\phi(q^3)^2\,\phi(q^{12})^2}\right), \tag{5.16}$$

and

$$\sum_{n\geq 0}\tilde{C}_n q^n=\frac{1}{2}\left(-\frac{\phi(q^2)}{\phi(q^4)}+\frac{\phi(q^6)^5}{\phi(q^2)\,\phi(q^3)^2\,\phi(q^{12})^2}\right). \tag{5.17}$$

Then $B'_n=\tilde{B}_{2n+1}$, $C'_n=\tilde{C}_{2n+1}$, $D'_n=\tilde{B}_{2n}$, and $F'_n=\tilde{C}_{2n+2}$. Therefore we have

$$\begin{aligned}
M\otimes I_{-2}\cong\sum_{k\geq 0}\Bigg\{&\Bigg(\sum_{m+n=k}K_{2m}A'_n+\sum_{m+n+2=k}K_{2m+3}E'_n\Bigg)\\
&\times V(-4\alpha_{-1}-\alpha_0-k\delta)\\
&\oplus\Bigg(\sum_{m+n=k}K_{2m}B'_n+\sum_{m+n+1=k}K_{2m+3}D'_n\Bigg)\\
&\times V(-4\alpha_{-1}-2\alpha_0-k\delta)\\
&\oplus\Bigg(\sum_{m+n=k}K_{2m}C'_n+\sum_{m+n+2=k}K_{2m+3}F'_n\Bigg)\\
&\times V(-4\alpha_{-1}-(k+1)\delta)\Bigg\}
\end{aligned}$$

$$\cong \sum_{k \geqslant 0} \left\{ \left(\sum_{m \geqslant 0} (K_{2m} \tilde{A}_{2k-2m} + K_{2m+3} \tilde{A}_{2k-2m-3}) \right) \right.$$
$$\times V(-4\alpha_{-1} - \alpha_0 - k\delta)$$
$$\oplus \left(\sum_{m \geqslant 0} (K_{2m} \tilde{B}_{2k-2m+1} + K_{2m+3} \tilde{B}_{2k-2m-2}) \right)$$
$$\times V(-4\alpha_{-1} - 2\alpha_0 - k\delta)$$
$$\oplus \left(\sum_{m \geqslant 0} (K_{2m} \tilde{C}_{2k-2m+1} + K_{2m+3} \tilde{C}_{2k-2m-2}) \right)$$
$$\left. \times V(-4\alpha_{-1} - (k+1)\delta) \right\}$$
$$= \sum_{k \geqslant 0} \left\{ \left(\sum_{m=0}^{2k} K_m \tilde{A}_{2k-m} - \tilde{A}_{2k-1} \right) V(-4\alpha_{-1} - \alpha_0 - k\delta) \right.$$
$$\oplus \left(\sum_{m=0}^{2k+1} K_m \tilde{B}_{2k+1-m} - \tilde{B}_{2k} \right) V(-4\alpha_{-1} - 2\alpha_0 - k\delta)$$
$$\left. \oplus \left(\sum_{m=0}^{2k+1} K_m \tilde{C}_{2k+1-m} - \tilde{C}_{2k} \right) V(-4\alpha_{-1} - (k+1)\delta) \right\}.$$

Let

$$P_k = \sum_{m=0}^{2k} K_m \tilde{A}_{2k-m} - \tilde{A}_{2k-1},$$
$$Q_k = \sum_{m=0}^{2k+1} K_m \tilde{B}_{2k+1-m} - \tilde{B}_{2k} - \delta_{k,1},$$
$$R_k = \sum_{m=0}^{2k+1} K_m \tilde{C}_{2k+1-m} - \tilde{C}_{2k}.$$

Then

$$M \otimes I_{-2} / V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \cong \sum_{k \geqslant 0} \{ P_k V(-4\alpha_{-1} - \alpha_0 - k\delta)$$
$$\oplus Q_k V(-4\alpha_{-1} - 2\alpha_0 - \kappa\delta)$$
$$\oplus R_k V(-4\alpha_{-1} - (k+1)\delta) \}. \quad (5.18)$$

Let

$$\sum_{k \geqslant 0} \tilde{P}_k q^k = \left(\sum_{m \geqslant 0} K_m q^m \right) \left(\sum_{n \geqslant 0} \tilde{A}_n q^n \right) - q \sum_{n \geqslant 0} \tilde{A}_n q^n,$$

$$\sum_{k\geq 0} \tilde{Q}_k q^k = \Big(\sum_{m\geq 0} K_m q^m\Big)\Big(\sum_{n\geq 0} \tilde{B}_n q^n\Big) - q\sum_{n\geq 0} \tilde{B}_n q^n - q^3,$$

$$\sum_{k\geq 0} \tilde{R}_k q^k = \Big(\sum_{m\geq 0} K_m q^m\Big)\Big(\sum_{n\geq 0} \tilde{C}_n q^n\Big) - q\sum_{n\geq 0} \tilde{C}_n q^n.$$

Then $P_k = \tilde{P}_{2k}$, $Q_k = \tilde{Q}_{2k+1}$, and $R_k = \tilde{R}_{2k+1}$. By (5.5), (5.6), (5.16), and (5.17), we have

$$\sum_{k\geq 0} \tilde{P}_k q^k = \frac{\phi(q^2)\,\phi(q^3)\,\phi(q^{12})}{\phi(q)\,\phi(q^4)\,\phi(q^6)}\left(\frac{\phi(q^2)^2}{\phi(q)\,\phi(q^4)} - q\right), \tag{5.19}$$

$$\sum_{k\geq 0} \tilde{Q}_k q^k = \frac{1}{2}\left(\frac{\phi(q^2)}{\phi(q^4)} + \frac{\phi(q^6)^5}{\phi(q^2)\,\phi(q^3)^2\,\phi(q^{12})^2}\right)\left(\frac{\phi(q^2)^2}{\phi(q)\,\phi(q^4)} - q\right) - q^3, \tag{5.20}$$

$$\sum_{k\geq 0} \tilde{R}_k q^k = \frac{1}{2}\left(-\frac{\phi(q^2)}{\phi(q^4)} + \frac{\phi(q^6)^5}{\phi(q^2)\,\phi(q^3)^2\,\phi(q^{12})^2}\right)\left(\frac{\phi(q^2)^2}{\phi(q)\,\phi(q^4)} - q\right). \tag{5.21}$$

We summarize the above calculation in the following proposition.

PROPOSITION 5.2. *Let*

$$\sum_{m\geq 0} P_m q^m = \sum_{m\geq 0} \tilde{P}_{2m} q^m,$$

$$\sum_{m\geq 0} Q_m q^m = \sum_{m\geq 0} \tilde{Q}_{2m+1} q^m,$$

and

$$\sum_{m\geq 0} R_m q^m = \sum_{m\geq 0} \tilde{R}_{2m+1} q^m,$$

where the $\tilde{P}_n$, $\tilde{Q}_n$, *and* $\tilde{R}_n$ *are defined by* (5.19)–(5.21). *Then we have*

$$\begin{aligned} I_{-4} \cong \sum_{m\geq 0} \{&P_m V(-4\alpha_{-1} - \alpha_0 - m\delta) \oplus Q_m V(-4\alpha_{-1} - 2\alpha_0 - m\delta) \\ &\oplus R_m V(-4\alpha_{-1} + (m+1)\delta)\} \oplus \Lambda^2(I_{-2}). \end{aligned}$$

Now we decompose the subspace I_{-5}. By (3.11), we have

$$\begin{aligned} I_{-5} &\cong V\otimes I_{-4}/V\otimes \Lambda^2(I_{-2}) \\ &\cong V\otimes (I_{-4}/\Lambda^2(I_{-2})). \end{aligned}$$

Hence by Proposition 4.2 and Proposition 5.2, we have

$$\begin{aligned} I_{-5} \cong V\otimes \sum_{m\geq 0} \{&P_m V(-4\alpha_{-1} - \alpha_0 - m\delta) \\ &\oplus Q_m V(-4\alpha_{-1} - 2\alpha_0 - m\delta) \oplus R_m V(-4\alpha_{-1} - (m+1)\delta)\} \end{aligned}$$

$$\begin{aligned}
&\cong V(-\alpha_{-1}) \otimes \sum_{m \geq 0} \{P_m V(-4\alpha_{-1} - \alpha_0 - m\delta) \\
&\qquad \oplus Q_m V(-4\alpha_{-1} - 2\alpha_0 - m\delta) \oplus R_m V(-4\alpha_{-1} - (m+1)\delta)\} \\
&\cong \sum_{m \geq 0} \Big\{ P_m \sum_{n \geq 0} \{A_n'' V(-5\alpha_{-1} - \alpha_0 - (m+n)\delta) \\
&\qquad \oplus B_n'' V(-5\alpha_{-1} - 2\alpha_0 - (m+n)\delta) \\
&\qquad \oplus C_n'' V(-5\alpha_{-1} - (m+n+1)\delta)\} \\
&\qquad \oplus Q_m \sum_{n \geq 0} \{D_n'' V(-5\alpha_{-1} - 2\alpha_0 - (m+n)\delta) \\
&\qquad \oplus E_n'' V(-5\alpha_{-1} - \alpha_0 - (m+n+1)\delta) \\
&\qquad \oplus F_n'' V(-5\alpha_{-1} - (m+n+2)\delta)\} \\
&\qquad \oplus R_m \sum_{n \geq 0} \{G_n'' V(-5\alpha_{-1} - (m+n+1)\delta) \\
&\qquad \oplus H_n'' V(-5\alpha_{-1} - \alpha_0 - (m+n+1)\delta) \\
&\qquad \oplus J_n'' V(-5\alpha_{-1} - 2\alpha_0 - (m+n+1)\delta)\} \Big\},
\end{aligned} \tag{5.22}$$

where

$$A_n'' = \sum_{j \in \mathbf{Z}} (p(n - 3j(14j+1)) - p(n - (6j+3)(7j+3))),$$

$$B_n'' = \sum_{j \in \mathbf{Z}} (p(n - 3j(14j-3)) - p(n - (6j+3)(7j+5))),$$

$$C_n'' = \sum_{j \in \mathbf{Z}} (p(n - 1 - 3j(14j+5)) - p(n - 1 - (6j+3)(7j+1))),$$

$$D_n'' = \sum_{j \in \mathbf{Z}} (p(n - j(42j+5)) - p(n - (6j+5)(7j+5))),$$

$$E_n'' = \sum_{j \in \mathbf{Z}} (p(n - (3j+1)(14j+1)) - p(n - (2j+1)(21j+16))),$$

$$F_n'' = \sum_{j \in \mathbf{Z}} (p(n - (2j+1)(21j+4)) - p(n - (3j+1)(14j+9))),$$

$$G_n'' = \sum_{j \in \mathbf{Z}} (p(n - j(42j+1)) - p(n - (6j+1)(7j+1))),$$

$$H_n'' = \sum_{j \in \mathbf{Z}} (p(n - j(42j-11)) - p(n - 6j+1)(7j+3))),$$

$$J_n'' = \sum_{j \in \mathbf{Z}} (p(n - 1 - (2j+1)(21j+1)) - p(n - 1 - (3j+2)(14j+3))).$$

Thus

$$\sum_{n\geqslant 0} A''_n q^n = \sum_{j\in\mathbf{Z}} (q^{3j(14j+1)} - q^{3(2j+1)(7j+3)}) \left(\sum_{m\geqslant 0} p(m) q^m\right)$$
$$= \left(\sum_{k\in\mathbf{Z}} (-1)^k q^{3k(7k+1)/2}\right) \frac{1}{\phi(q)}.$$

Since

$$\sum_{k\in\mathbf{Z}} (-1)^k q^{k(7k+1)/2} = \phi(q^7) \prod_{n\equiv \pm 3 \pmod 7} (1-q^n),$$

we have

$$\sum_{n\geqslant 0} A''_n q^n = \frac{\phi(q^{21})}{\phi(q)} \prod_{n\equiv \pm 9 \pmod{21}} (1-q^n). \tag{5.23}$$

Similarly, we obtain

$$\sum_{n\geqslant 0} B''_n q^n = \frac{\phi(q^{21})}{\phi(q)} \prod_{n\equiv \pm 6 \pmod{21}} (1-q^n), \tag{5.24}$$

$$\sum_{n\geqslant 0} C''_n q^n = q\frac{\phi(q^{21})}{\phi(q)} \prod_{n\equiv \pm 3 \pmod{21}} (1-q^n), \tag{5.25}$$

$$\sum_{n\geqslant 0} D''_n q^{3n} = \left(\sum_{k\equiv 0,\,1 \pmod 6} (-1)^k q^{k(7k+5)/2}\right) \frac{1}{\phi(q^3)}, \tag{5.26}$$

$$\sum_{n\geqslant 0} E''_n q^{3n+2} = \left(\sum_{k\equiv 1,\,2 \pmod 6} (-1)^{k+1} q^{k(7k+3)/2}\right) \frac{1}{\phi(q^3)}, \tag{5.27}$$

$$\sum_{n\geqslant 0} F''_n q^{3n+3} = \left(\sum_{k\equiv 2,\,3 \pmod 6} (-1)^k q^{k(7k+1)/2}\right) \frac{1}{\phi(q^3)}, \tag{5.28}$$

$$\sum_{n\geqslant 0} G''_n q^{3n} = \left(\sum_{k\equiv 0,\,5 \pmod 6} (-1)^k q^{k(7k+1)/2}\right) \frac{1}{\phi(q^3)}, \tag{5.29}$$

$$\sum_{n\geqslant 0} H''_n q^{3n+2} = \left(\sum_{k\equiv 4,\,5 \pmod 6} (-1)^{k+1} q^{k(7k+3)/2}\right) \frac{1}{\phi(q^3)}, \tag{5.30}$$

$$\sum_{n\geqslant 0} J''_n q^{3n+3} = \left(\sum_{k\equiv 3,\,4 \pmod 6} (-1)^k q^{k(7k+5)/2}\right) \frac{1}{\phi(q^3)}. \tag{5.31}$$

Let

$$\sum_{n\geqslant 0} \tilde{D}_n q^n = \frac{\phi(q^7)}{\phi(q^3)} \prod_{n\equiv \pm 1 \pmod 7} (1-q^n), \tag{5.32}$$

$$\sum_{n\geqslant 0} \tilde{E}_n q^n = -\frac{\phi(q^7)}{\phi(q^3)} \prod_{n\equiv \pm 2 \pmod 7} (1-q^n), \tag{5.33}$$

and

$$\sum_{n\geq 0} \tilde{F}_n q^n = \frac{\phi(q^7)}{\phi(q^3)} \prod_{n\equiv \pm 3 \pmod 7} (1-q^n). \tag{5.34}$$

Then it is easy to see that $D''_n + J''_{n-1} = \tilde{D}_{3n}$, $E''_n + H''_n = \tilde{E}_{3n+2}$, and $F''_{n-1} + G''_n = \tilde{F}_{3n}$. Therefore by (5.22), we obtain:

PROPOSITION 5.3. *Define*

$$\begin{aligned}\sum_{m\geq 0} U_m q^m = &\Big(\sum_{n\geq 0} P_n q^n\Big)\Big(\sum_{n\geq 0} A''_n q^n\Big)\\ &+q\Big(\sum_{n\geq 0} Q_n q^n\Big)\Big(\sum_{n\geq 0} E''_n q^n\Big)\\ &+q\Big(\sum_{n\geq 0} R_n q^n\Big)\Big(\sum_{n\geq 0} H''_n q^n\Big),\end{aligned} \tag{5.35}$$

$$\begin{aligned}\sum_{m\geq 0} V_m q^m = &\Big(\sum_{n\geq 0} P_n q^n\Big)\Big(\sum_{n\geq 0} B''_n q^n\Big)\\ &+\Big(\sum_{n\geq 0} Q_n q^n\Big)\Big(\sum_{n\geq 0} D''_n q^n\Big)\\ &+q\Big(\sum_{n\geq 0} R_n q^n\Big)\Big(\sum_{n\geq 0} J''_n q^n\Big),\end{aligned} \tag{5.36}$$

and

$$\begin{aligned}\sum_{m\geq 0} W_m q^m = &\Big(\sum_{n\geq 0} P_n q^n\Big)\Big(\sum_{n\geq 0} C''_n q^n\Big)\\ &+q\Big(\sum_{n\geq 0} Q_n q^n\Big)\Big(\sum_{n\geq 0} F''_n q^n\Big)\\ &+\Big(\sum_{n\geq 0} R_n q^n\Big)\Big(\sum_{n\geq 0} G''_n q^n\Big).\end{aligned} \tag{5.37}$$

Then we have

$$\begin{aligned}I_5 \cong \sum_{m\geq 0} \{&U_m V(-5\alpha_{-1} - \alpha_0 - m\delta)\\ &\oplus V_m V(-5\alpha_{-1} - 2\alpha_0 - m\delta) \oplus W_m V(-5\alpha_{-1} - (m+1)\delta)\}.\end{aligned}$$

6. Root Multiplicities

To compute the root multiplicities of $HA_1^{(1)}$, we need to compute the weight multiplicities of integrable highest weight modules over $A_1^{(1)}$. For this purpose, we introduce part of representation theory of $U_q\widehat{sl}(n)$ developed in [DJKMO] and [JMMO]. We will introduce $A_1^{(1)}$ theory only, since that is what we need here.

Definition 6.1. An *extended Young diagram* is a sequence $(y_k)_{k\geqslant 0}$ such that (i) $y_k \in \mathbf{Z}$, $y_k \leqslant y_{k+1}$ for all $k \geqslant 0$, (ii) there exists fixed $y_\infty \in \mathbf{Z}$ such that $y_k = y_\infty$ for all $k \gg 0$. The integer y_∞ is called the *charge* of Y.

Thus an extended Young diagram $Y = (y_k)_{k \geqslant 0}$ is an infinite Young diagram drawn on the lattice in the right half plane with sited $\{(i, j) \in \mathbf{Z} \times \mathbf{Z} \mid i \geqslant 0\}$, where y_k denotes the depth of the kth column. In Fig. 6.1 below, we illustrate the extended Young diagram $Y = (-3, -2, -2, 0, 1, 1, ...)$ of charge 1.

In the sequel, we will consider extended Young diagrams with charges 0 and 1 only, and each extended Young diagram will be regarded as a weight vector for $A_1^{(1)}$. We define the *weight* of an extended Young diagram as follows. We color the nodes of each extended Young diagram white and black alternatingly, the node at he left-top corner being white if Y has charge 0 and being black if Y has charge 1. If Y is an extended Young diagram with charge 0 and has n_0 white nodes and n_1 black nodes, we define the weight of Y to be

$$\mathrm{wt}(Y) = \Lambda_0 - n_0\alpha_0 - n_1\alpha_1. \tag{6.1}$$

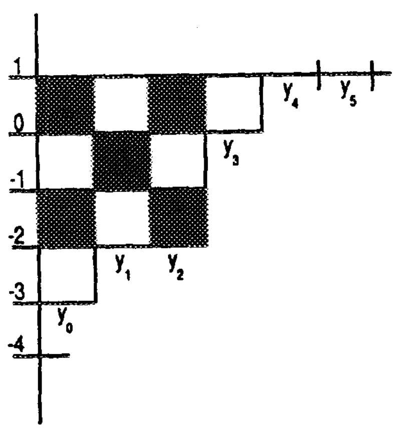

Figure 6.1

If Y is an extended Young diagram with charge 1 and has n_0 white nodes and n_1 black nodes, we define the weight of Y to be

$$\mathrm{wt}(Y) = \Lambda_1 - n_0\alpha_0 - n_1\alpha_1. \tag{6.2}$$

Thus the weight of the extended Young diagram in Figure 6.1 is $\Lambda_1 - 6\alpha_0 - 5\alpha_1$.

DEFINITION 6.2. Fix a positive integer l. We define a *pattern* to be a map

$$t: \mathbf{Z} \times \mathbf{Z}_{k \geqslant 0} \to \mathbf{Z}$$
$$(j, k) \mapsto t_{jk}$$

such that (i) for all j, $(t_{jk})_{k \geqslant 0}$ is an extended Young diagram, (ii) $t_{jk} \leqslant t_{j+1,k}$ for all j, k, (iii) $t_{j+l,k} = t_{jk} + 2$ for all j, k. We say that the pattern t is normalized if $0 \leqslant \gamma_1 \leqslant \cdots \leqslant \gamma_l < 2$, where $\gamma_j = t_{j\infty}$ is the charge of the extended Young diagram $(t_{jk})_{k \geqslant 0}$. We call $\gamma = (\gamma_1, ..., \gamma_l)$ the charge of t. We identify the pattern t with a sequence $\mathbf{Y} = (Y_j)_{j \in \mathbf{Z}}$ of extended Young diagrams $Y_j = (t_{jk})_{k \geqslant 0}$. Let $\mathscr{T}$ denote the set of all patterns.

For a fixed dominant integral weight $\Lambda = \Lambda_{\gamma_1} + \cdots + \Lambda_{\gamma_l}$ with $0 \leqslant \gamma_1 \leqslant \cdots \leqslant \gamma_l < 2$, defined the set

$$\mathscr{Y}(\Lambda) = \{\mathbf{Y} = (Y_j)_{j \in \mathbf{Z}} \in \mathscr{T} \mid \mathbf{Y} \text{ has charge } \gamma = (\gamma_1, ..., \gamma_l)\}.$$

Note that $\mathbf{Y} = \mathscr{Y}(\Lambda)$ is completely determined by $(Y_1, ..., Y_l)$ using the periodicity with respect to j with period l. So we will identify $\mathbf{Y}$ with $(Y_1, ..., Y_l)$. The weight of $\mathbf{Y}$ is defined to be

$$\mathrm{wt}(\mathbf{Y}) = \mathrm{wt}(Y_1) + \cdots + \mathrm{wt}(Y_l). \tag{6.3}$$

Let $\mathscr{H}(\Lambda)$ be the set of all $\mathbf{Y} \in \mathscr{Y}(\Lambda)$ such that

$$Y_1 \supset Y_2 \supset \cdots Y_l, \tag{6.4}$$

$$Y_l \supset Y_{l+1}, \tag{6.5}$$

$$\text{for each } k \geqslant 0, \text{ there exists some } j \text{ such that } t_{j+1,k} > t_{j,k+1}. \tag{6.6}$$

Define

$$Y(\Lambda, \alpha) = \#\{\mathbf{Y} \in \mathscr{H}(\Lambda) \mid \mathrm{wt}(\mathbf{Y}) = \alpha\}. \tag{6.7}$$

Then we have:

THEOREM 6.3 [DJKMO, JMMO].

$$\dim V(\Lambda)_\alpha = Y(\Lambda, \alpha).$$

Remark 6.4. This result was first obtained in [DJKMO]. In [JMMO], they give a simpler proof using the crystal bases of integrable irreducible representations of quantum affine algebra $U_q\widehat{sl}(n)$.

We are ready to state our main results.

THEOREM 6.5. *We have*

$$\dim(I_{-2})_\alpha = Y(-2\alpha_{-1}-\alpha_0, \alpha) = E\left(3-\frac{(\alpha|\alpha)}{2}\right),$$

where the function $E(n)$ *is defined by* (2.13),

$$\begin{aligned}\dim(I_{-3})_\alpha = \sum_{m\geq 0} (A_m Y(-3\alpha_{-1}-\alpha_0-m\delta, \alpha) \\ + B_m Y(-3\alpha_{-1}-(m+1)\delta, \alpha)),\end{aligned}$$

where the A_m *and* B_m *are defined in Proposition* 5.1,

$$\begin{aligned}\dim(I_{-4})_\alpha = \sum_{m\geq 0} \{&P_m Y(-4\alpha_{-1}-\alpha_0-m\delta, \alpha) \\ &+ Q_m Y(-4\alpha_{-1}-2\alpha_0-m\delta, \alpha) \\ &+ R_m Y(-4\alpha_{-1}-(m+1)\delta, \alpha)\} \\ &+ B(\alpha, V(-2\alpha_{-1}-\alpha_0)) - \frac{1}{2}\varepsilon(\alpha, 2)\, E\left(3-\frac{(\alpha|\alpha)}{8}\right),\end{aligned}$$

where the P_m, Q_m, R_m *are defined in Proposition* 5.2 *and* $\varepsilon(\alpha, t)$ *is the signum function defined by* (2.12), *and*

$$\begin{aligned}\dim(I_{-5})_\alpha = \sum_{m\geq 0} \{&U_m Y(-5\alpha_{-1}-\alpha_0-m\delta, \alpha) \\ &+ V_m Y(-5\alpha_{-1}-2\alpha_0-m\delta, \alpha) \\ &+ W_m Y(-5\alpha_{-1}-(m+1)\delta, \alpha)\},\end{aligned}$$

where the U_m, V_m, W_m *are defined by* (5.35)–(5.37).

THEOREM 6.6. *Let* $L=\bigoplus_{n\in\mathbf{Z}} L_n$ *be the realization of the hyperbolic Kac–Moody Lie algebra* $HA_1^{(1)}$ *constructed in Section* 1. *Then we have*

$$\dim(L_0)_\alpha = 1,$$

$$\dim(L_{-1})_\alpha = p\left(1-\frac{(\alpha|\alpha)}{2}\right) = Y(-\alpha_{-1}, \alpha),$$

$$\dim(L_{-n})_\alpha = \dim(G_{-n})_\alpha - \dim(I_{-n})_\alpha \qquad \textit{for} \quad n=2, 3, 4, 5,$$

where $\dim(G_{-n})_\alpha$ *is given by the Witt formula* (2.8) *and* $\dim(I_{-n})_\alpha$ *is given by Theorem* 6.5.

We illustrate our root multiplicity formula in the following examples.

EXAMPLE 6.7. Consider the root $\alpha = -3\alpha_{-1} - 4\alpha_0 - 4\alpha_1$ of level 3. By Theorem 6.5, we have

$$\begin{aligned}\dim(I_{-3})_\alpha &= \sum_{m=0}^{3} (A_m Y(-3\alpha_{-1} - \alpha_0 - m\delta, \alpha) \\ &\quad + B_m Y(-3\alpha_{-1} - (m+1)\delta, \alpha)) \\ &= \sum_{m=0}^{3} A_m Y(\Lambda_0 + 2\Lambda_1, \Lambda_0 + 2\Lambda_1 - \alpha_1 - (3-m)\delta) \\ &\quad + \sum_{m=0}^{3} B_m Y(3\Lambda_0, 3\Lambda_0 - (3-m)\delta).\end{aligned}$$

By (5.2), (5.3), and Proposition 5.1, we have

$$\sum_{m \geqslant 0} A_m q^m = 1 + q + q^2 + 2q^3 + 3q^4 + 4q^5 + 6q^6 + \cdots,$$

and

$$\sum_{m \geqslant 0} B_m q^m = q + q^2 + 2q^3 + 2q^4 + 3q^5 + 3q^6 + \cdots.$$

We will compute $Y(3\Lambda_0, 3\Lambda_0 - 2\delta)$. Others can be computed in the same way. Since the weight is $3\Lambda_0 - 2\alpha_0 - 2\alpha_1$, the corresponding weight vector must have two white nodes and two black nodes. Now it is easy to check that the choices in Fig. 6.2 are all the possible choices satisfying (6.4)–(6.6).

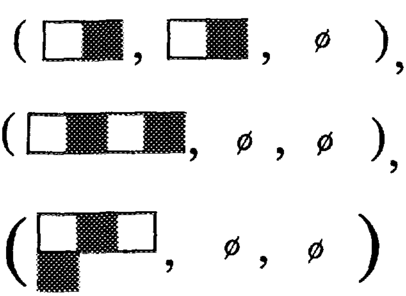

FIGURE 6.2.

481 160 2-16

Thus $Y(3\Lambda_0, 3\Lambda_0-2\delta)=3$. Similarly, we get

$$Y(3\Lambda_0, 3\Lambda_0)=1,$$
$$Y(3\Lambda_0, 3\Lambda_0-\delta)=1,$$
$$Y(3\Lambda_0, 3\Lambda_0-3\delta)=6,$$

and

$$Y(\Lambda_0+2\Lambda_1, \Lambda_0+2\Lambda_1-\alpha_1)=1,$$
$$Y(\Lambda_0+2\Lambda_1, \Lambda_0+2\Lambda_1-\alpha_1-\delta)=3,$$
$$Y(\Lambda_0+2\Lambda_1, \Lambda_0+2\Lambda_1-\alpha_1-2\delta)=6,$$
$$Y(\Lambda_0+2\Lambda_1, \Lambda_0+2\Lambda_1-\alpha_1-3\delta)=13.$$

Therefore,

$$\begin{aligned}\dim(I_{-3})_\alpha &= (1)(13)+(1)(6)+(1)(3)+(2)(1)+(0)(6)+(1)(3)\\ &\quad+(1)(1)+(2)(1)=30.\end{aligned}$$

In Example 2.3, we have seen that

$$\dim(G_{-3})_\alpha=35.$$

Hence we get

$$\dim(L_{-3})_\alpha=35-30=5.$$

EXAMPLE 6.8. In this example, we compute the multiplicity of the root $\alpha=-4\alpha_{-1}-5\alpha_0-4\alpha_1$. By Theorem 6.5, we have

$$\begin{aligned}\dim(I_{-4})_\alpha &= \sum_{m=0}^{4} P_m\, Y(-4\alpha_{-1}-\alpha_0-m\delta, \alpha)\\ &\quad+\sum_{m=0}^{3} Q_m\, Y(-4\alpha_{-1}-2\alpha_0-m\delta, \alpha)\\ &\quad+\sum_{m=0}^{3} R_m\, Y(-4\alpha_{-1}-(m+1)\,\delta, \alpha)\\ &\quad+B(\alpha, L(-2\alpha_{-1}-\alpha_0))-\frac{1}{2}\,\varepsilon(\alpha, 2)\, E\left(3-\frac{(\alpha\,|\,\alpha)}{2}\right)\end{aligned}$$

$$= \sum_{m=0}^{4} P_m Y(2\Lambda_0 + 2\Lambda_1, 2\Lambda_0 + 2\Lambda_1 - (4-m)\delta)$$
$$+ \sum_{m=0}^{3} Q_m Y(4\Lambda_1, 4\Lambda_1 - \alpha_1 - (3-m)\delta)$$
$$+ \sum_{m=0}^{3} R_m Y(4\Lambda_0, 4\Lambda_0 - \alpha_0 - (3-m)\delta)$$
$$+ B(-4\alpha_{-1} - 5\alpha_0 - 4\alpha_1, L(-2\alpha_{-1} - \alpha_0)).$$

By (5.19)–(5.21) and Proposition 5.2, we have

$$\sum_{m \geq 0} P_m q^m = \sum_{m \geq 0} \tilde{P}_{2m} q^m$$
$$= 1 + q + 4q^2 + 7q^3 + 16q^4 + 27q^5 + 49q^6 + \cdots,$$

$$\sum_{m \geq 0} Q_m q^m = \sum_{m \geq 0} \tilde{Q}_{2m+1} q^m$$
$$= q + 2q^2 + 5q^3 + 9q^4 + 17q^5 + \cdots,$$

$$\sum_{m \geq 0} R_m q^m = \sum_{m \geq 0} \tilde{R}_{2m+1} q^m$$
$$= q + 2q^2 + 5q^3 + 9q^4 + 17q^5 + \cdots.$$

By (6.4)–(6.6) and Theorem 6.3, we obtain

$$Y(2\Lambda_0 + 2\Lambda_1, 2\Lambda_0 + 2\Lambda_1) = 1,$$
$$Y(2\Lambda_0 + 2\Lambda_1, 2\Lambda_0 + 2\Lambda_1 - \delta) = 2,$$
$$Y(2\Lambda_0 + 2\Lambda_1, 2\Lambda_0 + 2\Lambda_1 - 2\delta) = 6,$$
$$Y(2\Lambda_0 + 2\Lambda_1, 2\Lambda_0 + 2\Lambda_1 - 3\delta) = 12,$$
$$Y(2\Lambda_0 + 2\Lambda_1, 2\Lambda_0 + 2\Lambda_1 - 4\delta) = 26,$$

$$Y(4\Lambda_1, 4\Lambda_1 - \alpha_1) = 1,$$
$$Y(4\Lambda_1, 4\Lambda_1 - \alpha_1 - \delta) = 2,$$
$$Y(4\Lambda_1, 4\Lambda_1 - \alpha_1 - 2\delta) = 5,$$
$$Y(4\Lambda_1, 4\Lambda_1 - \alpha_1 - 3\delta) = 10,$$

and

$$Y(4\Lambda_0, 4\Lambda_0 - \alpha_0) = 1,$$
$$Y(4\Lambda_0, 4\Lambda_0 - \alpha_0 - \delta) = 2,$$
$$Y(4\Lambda_0, 4\Lambda_0 - \alpha_0 - 2\delta) = 5,$$
$$Y(4\Lambda_0, 4\Lambda_0 - \alpha_0 - 3\delta) = 10.$$

In Example 2.5, we obtained

$$B(-4\alpha_{-1}-5\alpha_0-4\alpha_1, L(-2\alpha_{-1}-\alpha_0))=25.$$

Therefore,

$$\begin{aligned}\dim(I_{-4})_\alpha &= (1)(26)+(1)(12)+(4)(6)+(7)(2)+(16)(1)\\ &\quad +(0)(10)+(1)(5)+(2)(2)+(5)(1)+(0)(10)\\ &\quad +(1)(5)+(2)(2)+(5)(1)+25=145.\end{aligned}$$

In Example 2.3, we have seen that

$$\dim(G_{-4})_\alpha=150.$$

Hence

$$\dim(L_{-4})_\alpha=150-145=5.$$

EXAMPLE 6.9. In this example, we compute the multiplicity of $\alpha=-5\alpha_{-1}-6\alpha_0-5\alpha_1$. By Theorem 6.5, we have

$$\begin{aligned}\dim(I_{-5})_\alpha &= \sum_{m=0}^{5} U_m Y(-5\alpha_{-1}-\alpha_0-m\delta, \alpha)\\ &\quad +\sum_{m=0}^{4} V_m Y(-5\alpha_{-1}-2\alpha_0-m\delta, \alpha)\\ &\quad +\sum_{m=0}^{4} W_m Y(-5\alpha_{-1}-(m+1)\delta, \alpha)\\ &= \sum_{m=0}^{5} U_m Y(3\Lambda_0+2\Lambda_1, 3\Lambda_0+2\Lambda_1-(5-m)\delta)\\ &\quad +\sum_{m=0}^{4} V_m Y(\Lambda_0+4\Lambda_1, \Lambda_0+4\Lambda_1-\alpha_1-(4-m)\delta)\\ &\quad +\sum_{m=0}^{4} W_m Y(5\Lambda_0, 5\Lambda_0-\alpha_0-(4-m)\delta).\end{aligned}$$

By (5.35)–(5.37), we have

$$\sum_{m\geqslant 0} U_m q^m = 1+2q+8q^2+20q^3+51q^4+110q^5+\cdots,$$

$$\sum_{m\geqslant 0} V_m q^m = 1+3q+10q^2+24q^3+58q^4+\cdots,$$

$$\sum_{m\geqslant 0} W_m q^m = 2q+4q^2+13q^3+27q^4+\cdots.$$

By (6.4)–(6.6) and Theorem 6.3, we get

$$Y(3\Lambda_0+2\Lambda_1, 3\Lambda_0+2\Lambda_1)=1,$$
$$Y(3\Lambda_0+2\Lambda_1, 3\Lambda_0+2\Lambda_1-\delta)=2,$$
$$Y(3\Lambda_0+2\Lambda_1, 3\Lambda_0+2\Lambda_1-2\delta)=6,$$
$$Y(3\Lambda_0+2\Lambda_1, 3\Lambda_0+2\Lambda_1-3\delta)=13,$$
$$Y(3\Lambda_0+2\Lambda_1, 3\Lambda_0+2\Lambda_1-4\delta)=28,$$
$$Y(3\Lambda_0+2\Lambda_1, 3\Lambda_0+2\Lambda_1-5\delta)=54,$$
$$Y(\Lambda_0+4\Lambda_1, \Lambda_0+4\Lambda_1-\alpha_1)=1,$$
$$Y(\Lambda_0+4\Lambda_1, \Lambda_0+4\Lambda_1-\alpha_1-\delta)=3,$$
$$Y(\Lambda_0+4\Lambda_1, \Lambda_0+4\Lambda_1-\alpha_1-2\delta)=7,$$
$$Y(\Lambda_0+4\Lambda_1, \Lambda_0+4\Lambda_1-\alpha_1-3\delta)=16,$$
$$Y(\Lambda_0+4\Lambda_1, \Lambda_0+4\Lambda_1-\alpha_1-4\delta)=32,$$

and

$$Y(5\Lambda_0, 5\Lambda_0-\alpha_0)=1,$$
$$Y(5\Lambda_0, 5\Lambda_0-\alpha_0-\delta)=2,$$
$$Y(5\Lambda_0, 5\Lambda_0-\alpha_0-2\delta)=5,$$
$$Y(5\Lambda_0, 5\Lambda_0-\alpha_0-3\delta)=10,$$
$$Y(5\Lambda_0, 5\Lambda_0-\alpha_0-4\delta)=21.$$

Therefore

$$\begin{aligned}\dim(I_{-5})_\alpha &= (1)(54)+(2)(28)+(8)(13)+(20)(6)+(51)(2)\\ &\quad+(110)(1)+(1)(32)+(3)(16)+(10)(7)+(24)(3)\\ &\quad+(58)(1)+(0)(21)+(2)(10)+(4)(5)+(13)(2)\\ &\quad+(27)(1)=919.\end{aligned}$$

In Example 2.3, we have seen that

$$\dim(G_{-5})_\alpha=926.$$

Hence we obtain

$$\dim(L_{-5})_\alpha=926-919=7.$$

Remark 6.10. In [Fr], Frenkel conjectured that for a hyperbolic Kac–Moody Lie algebra $\mathfrak{g}$, we have

$$\dim \mathfrak{g}_\alpha \leqslant p^{(l-2)}\left(1 - \frac{(\alpha \mid \alpha)}{2}\right),$$

where the l is the size of the generalized Cartan matrix of $\mathfrak{g}$ and the function $p^{(l-2)}(n)$ is defined by

$$\sum_{n \geqslant 0} p^{(l-2)}(n)\, q^n = \frac{1}{\phi(q)^{l-2}} = \frac{1}{\prod_{n \geqslant 1} (1-q^n)^{l-2}}.$$

The root multiplicities computed in the above examples are compatible with his conjecture. Although this conjecture is not true for the hyperbolic Kac–Moody Lie algebra E_{10} [KMW], the numerical data indicate that it is still valid for $HA_1^{(1)}$.

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